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A deformed medium including a defect field and differential forms

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Abstract
We consider basic equations for a deformed medium including a defect field on the basis of differential forms. To make our analysis, we extend three basic equations: (I) an incompatibility equation; (II) the Peach–Köhler equation; (III) the Navier equation based on the Hodge duality of the deformed medium. By combining two exterior differential operators, we derive (I) an incompatibility equation that extends the compatibility equation to include a defect field. The Hodge dual of the incompatibility equation becomes a generalized stress function, which includes previously derived stress functions such as Beltrami's, Morera's, Maxwell's and Airy's stress functions. By applying homotopy operators, we extend (II) the Peach–Köhler equation to include disclinations. In this case, we can define the basic quantities of stress space by analogy with the monopole theory. By combining exterior differential operators and star operators, we extend (III) the Navier equation to include a defect field. In this analysis, we define a Navier operator that is related to the Laplace operator through Hodge duality. We consider gauge conditions for a defect field based on the differential geometry of a deformed medium. This suggests a duality between yielding and fatigue fractures. The gauge condition in strain space–time is interpreted as basic relations in polycrystalline plastic deformation.

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1. Introduction

A differential geometrical description of a deformed medium including a defect field was formulated by Kondo in 1952 [1]. Since then, this mathematical approach has been developed on the basis of Riemann–Cartan geometry [2–5] and applied in several fields such as the earth sciences [6, 7]. Following Kröner [3], we call this field the continuum theory of defects (CTD). The best known differential geometrical description of a physical phenomenon is the general theory of relativity, in which the gravitational potential is chosen as the space–time metric. In the CTD, there are two kinds of space–time—with different choices of metric—which describe a deformed medium [8–15]: one is called the strain space–time, for which the velocity distortion is chosen as the metric, and the other is called the stress space–time, for which the stress function potential is chosen as the metric.

Our previous study gave a combined analysis of these two space–times from the viewpoint of Hodge duality and derived several four-dimensional equations for defect fields in a systematic way [15]. For the analysis, we used a differential form and showed that the exterior differential operator can be used to derive continuity and kinematic equations for a defect field; we also showed that the Hodge star operator can be used to derive constitutive equations for a defect field [15]. Although the equations derived in [15] provide the basis for the CTD, they are inadequate for analysing applied physics problems such as yielding, fatigue fracturing and polycrystal plastic deformation. Hence, we attempt to derive/extend several practical equations: an incompatibility equation [3, 16, 17], the Peach–Köhler equation [18], the generalized Navier (and Laplace) equation [19] and the Golebiewska gauge conditions [4,20,21,33]. This is the main purpose of this paper. Moreover, we consider the relationship between the kind of equation used in the CTD and the type of corresponding operator in differential form.

This paper is structured as follows. In section 2, we show concisely how ordinary operators in differential form can be used to derive the basic equations in the CTD. In section 3, we derive an incompatibility equation and a generalized stress function by combining two exterior differential operators. In section 4, we derive Burgers and Frank vectors [22, 23] by using homotopy operators. Moreover, we consider the Peach–Köhler equation for disclinations. In section 5, we derive a Navier equation including the effect of defects by combining exterior differential operators and star operators. In section 6, we derive a harmonic equation for a defect field by using Laplace operators. For this derivation, we use the geometric expressions of the CTD [4]. These expressions naturally lead to Golebiewska gauge conditions in section 7. Section 8 is devoted to a discussion, in which we consider the mechanical interpretation of the Golebiewska gauge from the viewpoint of Kondo–Minagawa gauge theory [24] and the Taylor–Bishop–Hill (TBH) model [25, 26].

2. Continuity, and kinematic and constitutive equations

In this section, we give concisely the basic equations in the CTD based on [15]. Let $\{x^1, x^2, x^3, x^4\}$ be the Cartesian coordinates. In this study, we set $x^4 = ct = (E/\rho)^{1/2}t$, where t is the time variable, E is the elastic modulus and ρ is the density of mass [27]. The oriented volume element is given by $\mathrm{d}V = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, where the symbol $\wedge$ denotes the wedge product. The oriented surface element is given by the inner product: $\mathrm{d}s_A = \langle \partial_A, \mathrm{d}V \rangle$, where the index takes the values 1–3. The $(3+1)$-dimensional exterior differential operator is given by $\mathrm{d} = \mathrm{d}_s + \mathrm{d}t \wedge \partial_t$, where subscript s refers to pure space differentiation and subscript t to time differentiation. The p-form is denoted by Λ^p.

Let $\boldsymbol{B}$ and $\boldsymbol{S}$ be the strain and stress space–time, respectively. The quantities in the two spaces are summarized in tables 1 and 2. The most obvious difference between $\boldsymbol{B}$ and $\boldsymbol{S}$ is that the quantities in $\boldsymbol{B}$ such as strains, dislocations and disclinations are visible, while the quantities in $\boldsymbol{S}$ such as momentum, angular momentum and stresses are invisible. The continuity and kinematic equations in $\boldsymbol{B}$ are given by [4]

$$\mathrm{d}A^i = \Theta^i, \qquad \mathrm{d}\Theta^i = 0, \tag{1}$$

$$A^i = \mathrm{d}B^i + K^i, \qquad \Theta^i = \mathrm{d}K^i. \tag{2}$$

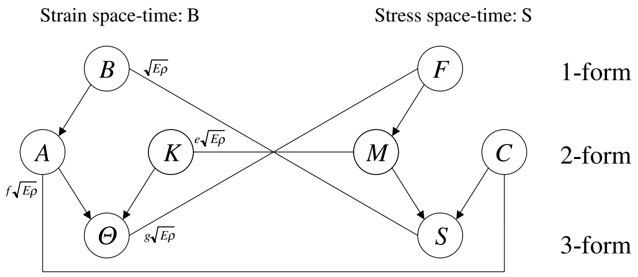

Figure 1. Hodge duality of a deformed medium including a defect field (after [15]). The key to the line types is as follows: →: $\Lambda^p \to \Lambda^{p+1}$ (continuity and kinematic equations); ——: $\Lambda^p \to \Lambda^{4-p}$ (constitutive equations).

Table 1. Quantities in strain space–time: $\boldsymbol{B}$ (after [15]).

	Time-like components	Space-like components
Distortion–velocity 1-form: $B^i = v^i\,\mathrm{d}t + \beta^i$	Velocity 0-form: v^i	Distortion 1-form: $\beta^i = \beta^i_A\,\mathrm{d}x^A$
Bend–twist–spin 2-form: $K^i = \omega^i \wedge \mathrm{d}t + \kappa^i$	Spin 1-form: $\omega^i = \omega^i_A\,\mathrm{d}x^A$	Bend–twist 2-form: $\kappa^i = \kappa^{iA}\,\mathrm{d}s_A$
Dislocation 2-form: $A^i = I^i \wedge \mathrm{d}t + \alpha^i$	Dislocation current 1-form: $I^i = I^i_A\,\mathrm{d}x^A$	Dislocation density 2-form: $\alpha^i = \alpha^{iA}\,\mathrm{d}s_A$
Disclination 3-form: $\Theta^i = J^i \wedge \mathrm{d}t + \theta^i$	Disclination current 2-form: $J^i = J^{iA}\,\mathrm{d}s_A$	Disclination density 3-form: $\theta^i = \vartheta^i\,\mathrm{d}V$

$\boldsymbol{S}$ can also be described by the following continuity and kinematic equations [8–15]:

$$\mathrm{d}M^i = S^i, \qquad \mathrm{d}S^i = 0, \tag{3}$$

$$M^i = \mathrm{d}F^i + C^i, \qquad S^i = \mathrm{d}C^i. \tag{4}$$

Continuity and kinematic equations in each space–time can be derived using the exterior differential operator $\mathrm{d} : \Lambda^p \to \Lambda^{p+1}$ described above. On the other hand, the constitutive equations can be derived using the linear Hodge operator $* : \Lambda^p(\boldsymbol{B}) \to \Lambda^{4-p}(\boldsymbol{S})$ as follows [15]:

$$\begin{aligned} S^i &= \sqrt{E\rho}\,{*B^i}, & M^i &= e\sqrt{E\rho}\,{*K^i}, \\ C^i &= f\sqrt{E\rho}\,{*A^i}, & F^i &= g\sqrt{E\rho}\,{*\Theta^i}. \end{aligned} \tag{5}$$

Equations (1)–(5) are four-dimensional expressions of previously derived basic equations in continuum mechanics (including the CTD). For instance, the first equation of (2) can be divided into time-like and space-like components: $\alpha^i = \mathrm{d}_s\beta^i + \kappa^i$, $I^i = \mathrm{d}_s v^i - \partial_t\beta^i + \omega^i$. The former is the well-known basic equation in the CTD [3] and the latter is Orowan's equation with spins. The time-like components of (3) are given by $\partial_t a^i + \mathrm{d}_s m^i = \sigma^i$ and $\partial_t P^i - \mathrm{d}_s\sigma^i = 0$. They are the conservation laws of angular momentum and momentum, respectively. The first equation of (5) can also be divided into time-like and space-like components: $p^i = \rho v^i$, $\sigma^{iA} = E\beta^{iA}$. The former is the constitutive equation for moments and the latter is the extended Hooke's law in the total strain theory of plasticity [28]. The results described above are summarized graphically in figure 1.

Table 2. Quantities in stress space–time: $\boldsymbol{S}$ (after [15]).

	Time-like components	Space-like components
Stress function 1-form $F^i = \phi^i\,\mathrm{d}t + \gamma^i$	Stress function 0-form: ϕ^i	Stress potential 1-form: $\gamma^i = \gamma^i_A\,\mathrm{d}x^A$
Couple-stress function 2-form: $C^i = c^i \wedge \mathrm{d}t + \delta^i$	Couple-stress function 1-form: $c^i = c^i_A\,\mathrm{d}x^A$	Couple-stress potential 2-form: $\delta^i = \delta^{iA}\,\mathrm{d}s_A$
Couple-stress 2-form: $M^i = m^i \wedge \mathrm{d}t + a^i$	Couple-stress 1-form $m^i = m^i_A\,\mathrm{d}x^A$	Angular momentum 2-form: $a^i = a^{iA}\,\mathrm{d}s_A$
Stress 3-form $S^i = \sigma^i \wedge \mathrm{d}t + P^i$	Stress 2-form: $\sigma^i = \sigma^{iA}\,\mathrm{d}s_A$	Momentum 3-form: $P^i = p^i\,\mathrm{d}V$

3. The incompatibility equation and the generalized stress function

All the continuity, kinematics and constitutive equations can be derived using the theory of differential forms in section 2. However, we did not derive a compatibility equation in strain space, which is necessary for strain components to have solutions of displacement components. Moreover, this compatibility equation should be enhanced to include the incompatibility tensor when defects exist [3,16,17]. In this section, we treat three-dimensional space in order to derive compatibility and incompatibility conditions in strain space. The incompatibility equations in strain space are given by the relation between the disclination densities and the distortions [3]:

$$\theta^i = \theta^i(\beta^i). \tag{6}$$

To derive this relation, we define the new disclination density by acting with the operator d_s from the right:

$$\widehat{\theta}^i = \alpha^i \overleftarrow{\mathrm{d}}_s \tag{7}$$

where the symbol '$\leftarrow$' means acting from the right. Substitution of the space-like components of (2) into this equation leads to

$$\eta^i = \mathrm{d}_s \beta^i \overleftarrow{\mathrm{d}}_s \tag{8}$$

where $\eta^i := \widehat{\theta}^i - \kappa^i \overleftarrow{\mathrm{d}}_s$ is the incompatibility tensor in the case of no extra matter except for dislocations and disclinations [17]. Now, one-to-one correspondences between operators in differential forms and in vector analysis are given by $\mathrm{d}\Lambda^0 \leftrightarrow \mathrm{grad}\,\Lambda^0$, $\mathrm{d}\Lambda^1 \leftrightarrow \mathrm{curl}\,\Lambda^1$ and $\mathrm{d}\Lambda^2 \leftrightarrow \mathrm{div}\,\Lambda^2$. By analogy with these, we set

$$\mathrm{d}\Lambda^0\overleftarrow{\mathrm{d}} \leftrightarrow \mathrm{grad}\,\Lambda^0\,\overleftarrow{\mathrm{grad}}, \qquad \mathrm{d}\Lambda^1\overleftarrow{\mathrm{d}} \leftrightarrow \mathrm{curl}\,\Lambda^1\,\overleftarrow{\mathrm{curl}} \qquad \mathrm{d}\Lambda^2\overleftarrow{\mathrm{d}} \leftrightarrow \mathrm{div}\,\Lambda^2\,\overleftarrow{\mathrm{div}}\,. \tag{9}$$

Note that acting twice from one side is not generally equivalent to acting once from each side ($\mathrm{curl}\,\mathrm{curl}(\cdots) \neq \mathrm{curl}(\cdots)\overleftarrow{\mathrm{curl}}$ and so on). In three-dimensional space, the 4-form $\mathrm{d}\Lambda^2\overleftarrow{\mathrm{d}}$ is identically zero. Since $\beta^i \in \Lambda^1$, equation (8) corresponds to

$$\eta = \mathrm{curl}\,\beta\,\overleftarrow{\mathrm{curl}}\,. \tag{10}$$

This is called the incompatibility equation in terms of strains [3, 16, 17], which satisfies the condition (6). In the particular case of $\eta = 0$, components of (10) are given by $\varepsilon_{ikl}\varepsilon_{jmn}\partial_k\partial_m\beta_{ln} = 0$. This is the well-known St Venant compatibility equation, which means that the topology of the continuum is invariant under elastic deformations [3]. However, when the deformation accompanies anelastic deformations such as emergences of defect fields, the topology of the continuum is not invariant, so the left-hand side of (10) is not zero.

Next, we rewrite the incompatibility equations (8) (or (10)) by using the constitutive equations (5) in three-dimensional space:

$$\sigma'^{i} = \mathrm{d}_s \phi'^{i} \overleftarrow{\mathrm{d}_s} + c'^{i} \overleftarrow{\mathrm{d}_s} \tag{11}$$

where we have introduced the dual stress space; that is, ϕ'^{i} is a dual stress function 1-form, c'^{i} is a dual couple-stress function 2-form and σ'^{i} is a dual stress 3-form. Because $\mathrm{d}_s\mathrm{d}_s = 0$ and the 4-form $\mathrm{d}_s c'^{i} \overleftarrow{\mathrm{d}_s}$ is identically zero in three-dimensional space, this equation gives $\mathrm{d}_s \sigma'^{i} = 0$ which is a continuity equation for stress. The vector analysis expression of (11) is $\sigma' = \mathrm{curl}\, \phi' \, \overleftarrow{\mathrm{curl}} + c' \, \overleftarrow{\mathrm{div}}$, which includes the previous stress functions as described below [16]. In the particular case of $c' = 0$, this corresponds to the equation for the Beltrami stress function: $\sigma_{ij} = \varepsilon_{ikl}\varepsilon_{jmn}\partial_k\partial_m\phi_{ln}$. The nondiagonal components of Beltrami stress functions are Morera stress functions and the diagonal components are Maxwell stress functions: $\sigma_{ij} = \delta_{ij}\partial_k\partial_k\phi - \partial_i\partial_j\phi$, where $\phi := \phi_{mm}$. Moreover, in the two-dimensional case the Maxwell stress functions are the well-known Airy stress functions: $\sigma_{11} = \partial_2\partial_2\phi$, $\sigma_{22} = \partial_1\partial_1\phi$ and $\sigma_{12} = \sigma_{21} = -\partial_1\partial_2\phi$. If we define the incompatibility tensor in stress space as $\eta^i(\boldsymbol{S}) := \sigma' - c' \, \overleftarrow{\mathrm{div}}$, equation (11) corresponds to (10). Therefore, the compatibility equation in stress space is given by $\varepsilon_{ikl}\varepsilon_{jmn}\partial_k\partial_m\phi_{ln} = 0$.

4. The Peach–Köhler equation and basic quantities

The force acting on a dislocation field is described by the Peach–Köhler equation, which is useful for solving various problems associated with deformations around dislocation lines [18]. However, this equation ignores the effect of the disclination field. Thus, in this section, we consider the equation for the force acting on a disclination field based on Amari's energy potential approach [14]. The Peach–Köhler equation is given by the product of the stress and Burgers vectors. Burgers vectors are considered basic quantities in defects dynamics [21]. In the analysis of basic quantities, a linear homotopy operator H plays an important role; it has the following properties [29]:

(I) $\mathrm{H} : \Lambda^p \to \Lambda^{p-1}$ for $p > 0$ and $\mathrm{H} : \Lambda^0 \to 0$;
(II) $\mathrm{dH} + \mathrm{Hd} =$ identity on Λ^p, $p > 0$;
(III) $\mathrm{HH} = 0$;
(IV) $\mathrm{HdH} = \mathrm{H}$, $\mathrm{dHd} = \mathrm{d}$.

By property (II), an arbitrary differential form X can be divided into two parts:

$$X = X_e + X_a \tag{12}$$

with

$$X_e := \mathrm{dH}X \qquad \text{and} \qquad X_a := \mathrm{Hd}X. \tag{13}$$

X_e is called the exact part of X, which is the kernel for d because $\mathrm{d}X_e = 0$. X_a is called the anti-exact part of X, which is the kernel for H because property (III) shows that $\mathrm{H}X_a = 0$. $X_a = \mathrm{Hd}X$ and (IV) shows that $\mathrm{d}X = \mathrm{d}X_a$ which means that a closed form ($\mathrm{d}X = 0$) is exact (Poincaré lemma).

Burgers vectors $b^i(\boldsymbol{B})$ are defined by the integral of distortions for the Burgers circle [30]. From Stokes's theorem and the property of the homotopy operator (II), this integral can be written as

$$b^i(\boldsymbol{B}) = \int_{\partial S} \beta^i = \int_{\partial S} (\mathrm{dH}\beta^i + \mathrm{Hd}\beta^i) = \int_S \mathrm{d}(\mathrm{dH}\beta^i + \mathrm{Hd}\beta^i) = \int_S \mathrm{dH}\,\mathrm{d}\beta^i. \tag{14}$$

In a similar fashion, we can define Frank vectors $f^i(\boldsymbol{B})$ as the integral of the bend–twists for the Frank surface:

$$f^i(\boldsymbol{B}) := \int_{\partial V} \kappa^i = \int_V \mathrm{dH}\, \mathrm{d}\kappa^i. \tag{15}$$

Using the space-like components of (2), relations (14) and (15) become well-known forms in the CTD [20, 21]:

$$b^i(\boldsymbol{B}) = \int_S (\alpha^i - \kappa^i), \tag{16}$$

$$f^i(\boldsymbol{B}) = \int_V \theta^i. \tag{17}$$

These relations show that basic quantities in strain space can be derived by the integration of a dislocation density 2-form and a disclination density 3-form. Now, in stress space, a couple-stress and a stress correspond to a dislocation and a disclination, respectively (see section 2). Thus, by analogy with (16) and (17), we integrate a couple-stress 1-form and a stress 2-form as follows:

$$\int_x (m^i - c^i) = \phi^i := b^i(\boldsymbol{S}), \tag{18}$$

$$\int_S \sigma^i = \int_{\partial S} c^i := f^i(\boldsymbol{S}), \tag{19}$$

where we use Stokes's theorem, equations (3) and (4) in three-dimensional space. The surface integral of the stress corresponds to an exerted force. Therefore, equations (18) and (19) show that stress functions and exerted forces are basic quantities in stress space just as Burgers and Frank vectors are basic quantities in strain space. In view of the constitutive equations (5) in three-dimensional space, these basic quantities are not independent of each other.

Next, we derive the force acting on a defect field on the basis of (16). Let us define the potential energy of distortions: $U(\beta) := \beta^i \wedge *_s\beta^i = E^{-1}\beta^i_A \wedge \sigma^{iA}\,\mathrm{d}V$, where $*_s : \Lambda^p \to \Lambda^{3-p}$ and we use (5). Since $U(\beta) \in \Lambda^3$ and $*_s\mathrm{d}V = 1$, we define the potential energy 0-form as $U'(\beta) := E^{-1}\beta^i_A \wedge \sigma^{iA}$. In this case, the force acting on a dislocation field is given by

$$F(\beta) = -\mathrm{d}_s U'(\beta) = -E^{-1}(\alpha^i_A - \kappa^i_A) \wedge \sigma^{iA} \tag{20}$$

where we use (2) and (3). This corresponds to the Peach–Köhler equation [18], because (16) shows that this force 1-form is equivalent to the wedge product of Burgers vectors and the stress. In a similar fashion, we have the force acting on a disclination field:

$$F(\kappa) = -\mathrm{d}_s U'(\kappa) = -(eE)^{-1}(\vartheta^{iA} \wedge m^i_A + \kappa^{iA} \wedge \sigma^i_A) \tag{21}$$

where $U'(\kappa) := (eE)^{-1}\kappa^{iA} \wedge m^i_A$. When $\kappa^{iA} \wedge \sigma^i_A = 0$, this equation corresponds to Amari's equation [14]. From (20) and (21), we obtain the force acting on the dislocations and disclinations: $F(\beta) + F(\kappa) = -E^{-1}(\alpha^i_A \wedge \sigma^{iA} + \vartheta^{iA} \wedge m^i_A)$, where we applied the normalization $e = 1$.

5. The Navier equation and the defect field

In the theory of elasticity, the deformed medium can be described by a Navier equation:

$$E_{ijkl}\partial_j\partial_{(l}u_{k)} - \rho\partial_t\partial_t u_i = F^B_i \tag{22}$$

where $\partial_{(l}u_{k)} := (\partial_l u_k + \partial_k u_l)/2$ and F^B_i is a body force. The Navier equation can be derived by combining two equations, i.e., the equation of motion: $\partial_j\sigma_{ij} + F^B_i = \partial_t p_i$ and the constitutive equations: $\sigma_{ij} = E_{ijkl}\beta_{kl}$ and $p_i = \rho v_i$. Moreover, we often use geometric relations, $\beta_{kl} = \partial_l u_k$ and $v_i = \partial_t u_i$, to derive the well-known form of the Navier equation (22). In the CTD, the equation of motion and the constitutive equations are extended like the second equation of (3) and the first equation of (5), respectively. Moreover, the geometric relation is extended to include the anti-exact part: $B^i = \mathrm{d}u^i + \mathrm{Hd}B^i$, where $u^i := \mathrm{H}B^i$ is an ordinary displacement 0-form. Therefore, the Navier equation in the CTD (without a body force) can be derived by combining the second equation of (3), the first equation of (5) and the extended geometric relations:

$$\sqrt{E\rho}\,\mathrm{d}{*}\mathrm{d}u^i = -\sqrt{E\rho}\,\mathrm{d}{*}\mathrm{Hd}B^i. \tag{23}$$

Equation (22) describes the displacement field due to the body force. On the other hand, the right-hand side of (23) is the anti-exact part of the distortion, which causes a dislocation field [21]. Therefore, equation (23) can be recognized as the Navier equation for a displacement field due to dislocations. In a similar fashion, we can derive the Navier equation for a rotational displacement field $r^i := \mathrm{H}K^i$ as follows:

$$e\sqrt{E\rho}\,\mathrm{d}{*}\mathrm{d}r^i = -e\sqrt{E\rho}\,\mathrm{d}{*}\mathrm{Hd}K^i + S^i \tag{24}$$

where we use the first equation of (3), the second equation of (5) and $K^i = \mathrm{d}r^i + \mathrm{Hd}K^i$. Because disclinations are caused by the anti-exact part of the bend–twist, $\mathrm{Hd}K^i$, this equation describes a displacement field due to disclinations (and stresses).

The results given above can be generalized as in the following analysis. The basic operators in differential form are exterior differential operators and Hodge star operators, with the derived kinematic equations and constitutive equations, respectively (see section 2). Thus, we operate with d and $*$ on an arbitrary p-form X in n-dimensional space to derive generalized kinematic and constitutive equations:

$$\mathrm{d}X = Y, \qquad *X = Z, \tag{25}$$

where Y is a $(p+1)$-form and Z is an $(n-p)$-form. Moreover, we use homotopy operators to derive generalized geometric relations:

$$\begin{aligned} X &= \mathrm{dH}X + \mathrm{Hd}X = \mathrm{d}x + y, \\ Y &= \mathrm{dH}Y + \mathrm{Hd}Y = \mathrm{d}y, \\ Z &= \mathrm{dH}Z + \mathrm{Hd}Z = \mathrm{d}z + \mathrm{Hd}Z, \end{aligned} \tag{26}$$

where $x := \mathrm{H}X$, $y := \mathrm{H}Y$ and $z := \mathrm{H}Z$. The first two relations show that $\mathrm{d}X = \mathrm{d}y$ although $X \neq y$. This property plays an important role in the gauge theory of defects (see section 7). The third relation generalizes the geometric relations, such as $B^i = \mathrm{d}u^i + \mathrm{Hd}B^i$ and $K^i = \mathrm{d}r^i + \mathrm{Hd}K^i$. By combining (25) and (26), we obtain

$$\mathrm{N}Z = \mathrm{d}G, \tag{27}$$

where $\mathrm{N} := \mathrm{d}{*}\mathrm{d}$ and $G := (-1)^{p(n-p)}\,\mathrm{Hd}X - {*}\mathrm{Hd}Z$. Equation (27) is the general form of the Navier equation—that is, when we recognize z as the ordinary and rotational displacements, equation (27) corresponds to (23) and (24), respectively. Thus, in this paper, we call $\mathrm{N} = \mathrm{d}{*}\mathrm{d}$ the Navier operator. It is clear that $\mathrm{NN} = 0$, $\mathrm{dN} = 0$ and $\mathrm{Nd} = 0$. In the next section, we derive the Laplace equation for the defect field by using this operator, N.

6. The Laplace equation for a defect field

In this section, we consider the Laplace equation for a defect field from the viewpoint of differential geometry. In four-dimensional space, the Laplace operator Δ is given by

$\Delta = \mathrm{d}\delta + \delta\mathrm{d} := \{\mathrm{d}, \delta\}$, where $\delta := *\mathrm{d}*$ [31]. It follows from the relation $\mathrm{N} = \mathrm{d}*\mathrm{d}$ that $\mathrm{d}\delta = \mathrm{N}*$ and $\delta\mathrm{d} = *\mathrm{N}$. This means that the Laplace operator is related to the Navier operator through the Hodge star operator:

$$\Delta = \{\mathrm{N}, *\}. \tag{28}$$

This relation shows that if arbitrary functions X satisfy $*X = 0$ and $\mathrm{N}X = 0$, they form Laplace fields: $\Delta X = 0$. The continuity and constitutive equations (1)–(5) give the Navier equation for dislocations and disclination fields as follows: $\mathrm{N}\Theta^i = 0$, $\mathrm{N}A^i = (g(E\rho)^{1/2})^{-1}(M^i - C^i)$, $\mathrm{N}*\Theta^i = g^{-1}(f - e)\Theta^i$ and $\mathrm{N}*A^i = f^{-1}(A^i - K^i)$. Therefore, operating with Δ on the defect field gives

$$\begin{aligned} \Delta\Theta^i &= \{\mathrm{N}, *\}\Theta^i = g^{-1}(f - e)\Theta^i, \\ \Delta A^i &= \{\mathrm{N}, *\}A^i = g^{-1}(fA^i - eK^i) + f^{-1}(A^i - K^i). \end{aligned} \tag{29}$$

This shows that the disclination field forms a Laplace field when the coefficients of two constitutive equations for a 2-form are equivalent (i.e. $e = f$) (see figure 1). Moreover, the disclination field also forms a Laplace field in the particular case where $g^{-1}f + f^{-1} = 0$. Next, we consider the geometrical meaning of the relations among the coefficients.

For this analysis, we use the differential geometrical description of a deformed medium including a defect field [4]. It shows that the physical quantities in the strain space–time can be expressed in terms of geometrical objects:

$$B^i = \psi^i, \qquad K^i = \Gamma^i_j \wedge \psi^j, \qquad A^i = T^i, \qquad \Theta^i = R^i{}_j \wedge \psi^j - \Gamma^i_j \wedge T^j, \tag{30}$$

where ψ^i is a dual basis 1-form, Γ^i_j is a connection 1-form, T^i is a torsion 2-form and $R^i{}_j$ is a curvature 2-form. Therefore, continuity equations (1) can be rewritten as

$$\mathrm{D}T^i = R^i{}_j \wedge \psi^j \qquad \text{and} \qquad \mathrm{D}R^i{}_j = 0 \tag{31}$$

where D is a covariant exterior differential operator which gives $\mathrm{D}T^i = \mathrm{d}T^i + \Gamma^i{}_j \wedge T^j$ and $\mathrm{D}R^i{}_j = \mathrm{d}R^i{}_j + \Gamma^i{}_k \wedge R^k{}_j - R^i{}_k \wedge \Gamma^k{}_j$. Equations (31) are the first and second Bianchi identities in the language of differential forms, respectively [31]. Moreover, the kinematics equations (2) become

$$T^i = \mathrm{D}\psi^i \qquad \text{and} \qquad R^i{}_j = \mathrm{D}\Gamma^i{}_j \tag{32}$$

where we use $\mathrm{D}\psi^i = \mathrm{d}\psi^i + \Gamma^i{}_j\psi^j$ and $\mathrm{D}\Gamma^i{}_j = \mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j$. Equations (32) are Cartan structure equations in the language of differential forms [31]. By analogy with (30), let us express the physical quantities of the stress space–time in terms of geometrical objects as follows [15]: $F^i = \psi'^i$, $C^i = \Gamma'^i{}_j \wedge \psi'^j$, $M^i = T'^i$, $S^i = R'^i{}_j \wedge \psi'^j - \Gamma'^i{}_j \wedge T'^j$. In this case, the continuity and kinematics equations in stress space–time (equations (3) and (4)) can be interpreted geometrically as Bianchi identities and Cartan structure equations in the stress space–time, respectively. Moreover, constitutive equations (5) can be rewritten as equations that describe the interaction between geometric objects in the strain space–time and those in the stress space–time:

$$\begin{aligned} R'^i{}_j \wedge \psi'^j - \Gamma'^i{}_j \wedge T'^j &= \sqrt{E\rho}\, *\psi^i, \\ T'^i &= e\sqrt{E\rho}\, *(\Gamma^i{}_j \wedge \psi^j), \\ \Gamma'^i{}_j \wedge \psi'^j &= f\sqrt{E\rho}\, *T^i, \\ \psi'^i &= g\sqrt{E\rho}\, *(R^i{}_j \wedge \psi^j - \Gamma^i{}_j \wedge T^j). \end{aligned} \tag{33}$$

Here, we consider the particular condition where geometric objects in the strain space–time are equivalent to corresponding objects in the stress space–time (i.e., $\psi^i = \psi'^i$, $\Gamma^i{}_j = \Gamma'^i{}_j$, $T^i = T'^i$ and $R^i{}_j = R'^i{}_j$), which means that the geometric structures of the two spaces are equivalent. In this case, equation (33) gives $g^{-1}f + e^{-1} = 0$ for $E\rho \neq 0$. This is equivalent to the condition that the defect field forms a Laplace field, i.e., $e = f$ and $g^{-1}f + f^{-1} = 0$. Therefore, it is found that the defect field forms a Laplace field in the particular case where geometric objects in strain and stress spaces are equivalent to each other.

7. Golebiewska gauge conditions

The gauge theory has played the main role in the mathematical description of defect fields [4, 5, 20, 21, 32–36]. For instance, Edelen and Golebiewska-Lasota have shown that the classical theory of defects admits a 45-fold Abelian gauge condition that is called the Golebiewska gauge [4, 20, 21, 33]. Previously the Golebiewska gauge has been studied in the strain space–time, so we consider a similar condition in the stress space–time on the basis of the geometrical approach of section 6.

Let us divide $\Gamma^i{}_j \wedge \psi^j$ into an exact part and an anti-exact part: $\Gamma^i{}_j \wedge \psi^j = (\Gamma^i{}_j \wedge \psi^j)_e + (\Gamma^i{}_j \wedge \psi^j)_a$. From the property of curvatures that measures the failure of covariant differentials to commute: $R^i{}_j \wedge \psi^j = \mathrm{DD}\psi^i$ and the Cartan structure equation: $T^i = \mathrm{D}\psi^i$, we have $R^i{}_j \wedge \psi^j = \mathrm{d}(\Gamma^i{}_j \wedge \psi^j) + \Gamma^i{}_j \wedge T^j$. Then, the above relation becomes

$$\mathrm{d}(\Gamma^i{}_j \wedge \psi^j)_a = R^i{}_j \wedge \psi^j - \Gamma^i{}_j \wedge T^j \tag{34}$$

where we use that the exact part is the kernel for d—that is, $\mathrm{d}(\Gamma^i{}_j \wedge \psi^j)_e = 0$. Equation (34) shows that the term $R^i{}_j \wedge \psi^j - \Gamma^i{}_j \wedge T^j$ is zero in the case of $(\Gamma^i{}_j \wedge \psi^j)_a = \mathrm{d}\xi^i$, where ξ^i is a 1-form. However, $(\Gamma^i{}_j \wedge \psi^j)_a$ is anti-exact, so $\mathrm{Hd}\xi^i = \xi^i_a = 0$, $\xi^i = \xi^i_e$ and $\mathrm{d}\xi^i = \mathrm{d}\xi^i_e = 0$. Then, we have $(\Gamma^i{}_j \wedge \psi^j)_a = 0$. In this case, $\Gamma^i{}_j \wedge \psi^j = (\Gamma^i{}_j \wedge \psi^j)_e = \mathrm{dH}(\Gamma^i{}_j \wedge \psi^j)$, so the Cartan structure equation $T^i = \mathrm{D}\psi^i$ becomes

$$T^i = \mathrm{d}\hat{\psi}^i = \mathrm{d}(\hat{\psi}^i_a) \tag{35}$$

with $\hat{\psi}^i := \psi^i + \mathrm{H}(\Gamma^i{}_j \wedge \psi^j)$. Equation (35) shows that T^i is zero in the case where $\hat{\psi}^i_a = \mathrm{d}\xi'^i$, where ξ'^i is a 0-form. In a similar fashion, to derive $(\Gamma^i{}_j \wedge \psi^j)_a = 0$, we can show that ξ'^i is an exact form and $\hat{\psi}^i_a = \psi^i_a + \mathrm{H}(\Gamma^i{}_j \wedge \psi^j) = 0$. The results given above can be summarized as follows: T^i and $R^i{}_j \wedge \psi^j - \Gamma^i{}_j \wedge T^j$ vanish in the case where $\psi^i_a = -\mathrm{H}(\Gamma^i{}_j \wedge \psi^j)$ and $(\Gamma^i{}_j \wedge \psi^j)_a = 0$—that is, $\psi^i = \psi^i_e - \mathrm{H}(\Gamma^i{}_j \wedge \psi^j)$ and $\Gamma^i{}_j \wedge \psi^j = (\Gamma^i{}_j \wedge \psi^j)_e$. On the basis of this result, we can show that T^i and $R^i{}_j \wedge \psi^j - \Gamma^i{}_j \wedge T^j$ are invariant under the following transformations:

$$\psi^i \to \psi^i + \psi^i_e - \mathrm{H}(\Gamma^i{}_j \wedge \psi^j) \qquad \text{and} \qquad \Gamma^i{}_j \wedge \psi^j \to \Gamma^i{}_j \wedge \psi^j + (\Gamma^i{}_j \wedge \psi^j)_e. \tag{36}$$

Next, we apply this geometric transformation to the strain and stress space–times.

Recall that quantities in the strain space–time can be expressed in terms of geometrical objects (see (30)). Therefore, equation (36) can be rewritten as follows: a dislocation 2-form and a disclination 3-form are invariant under the transformation

$$B^i \to B^i + \mathrm{dH}B^i - \mathrm{H}K^i \qquad \text{and} \qquad K^i \to K^i + \mathrm{dH}K^i. \tag{37}$$

This is called the Golebiewska gauge transformation in the strain space–time, and was derived first by Edelen [20]. In a similar fashion, the transformations (36) can be rewritten as follows: a couple-stress 2-form and a stress 3-form are invariant under the transformation

$$F^i \to F^i + \mathrm{dH}F^i - \mathrm{H}C^i \qquad \text{and} \qquad C^i \to C^i + C^i_e = C^i + \mathrm{dH}C^i. \tag{38}$$

Following (37), we call (38) the Golebiewska gauge transformation in the stress space–time. It is easy to show that the kinematics and continuity equations in the strain (or stress) space–time are also invariant under the transformations (37) (or (38)).

Next, we rewrite the transformations (37) and (38) by using constitutive equations. It follows from (5) and (38) that couple-stresses and stresses are invariant when dislocations and disclinations are transformed as follows:

$$A^i \to A^i - (f\sqrt{E\rho})^{-1} *\mathrm{dH}C^i \qquad \text{and} \qquad \Theta^i \to \Theta^i + (g\sqrt{E\rho})^{-1} *(\mathrm{dH}F^i - \mathrm{H}C^i). \tag{39}$$

Dislocations and disclinations are also invariant in the case where $\mathrm{H}C^i = \mathrm{dH}F^i$, for which the transformations (38) are trivial. Conversely, equations (5) and (37) indicate that dislocations and disclinations are invariant when couple-stresses and stresses are transformed as follows:

$$M^i \to M^i + e\sqrt{E\rho} *\mathrm{dH}K^i \qquad \text{and} \qquad S^i \to S^i + \sqrt{E\rho} *(\mathrm{dH}B^i - \mathrm{H}K^i). \tag{40}$$

Couple-stresses and stresses are also invariant in the trivial case where $\mathrm{H}K^i = \mathrm{dH}B^i$ (see (37)). In the next section, we discuss the mechanical meaning of the gauge transformations (39) and (40) on the basis of Kondo–Minagawa gauge theory [24] and the TBH model [25, 26].

8. Discussion and summary

In section 2, we showed that two operators d and $*$ play an important role in deriving the basic equations in the CTD. In the following section, we introduced some new operators in order to derive other basic equations in the CTD. The results are summarized as follows:

d:	continuity and kinematic equations
$*$:	constitutive equation
$\mathrm{d}(\cdots)\overleftarrow{\mathrm{d}}$:	incompatibility equation
$\mathrm{d}*(\cdots)\overleftarrow{\mathrm{d}}$:	generalized stress function
N:	Navier equation
$\{\mathrm{N}, *\} = \Delta$:	Laplace equation

where N is the Navier operator defined in section 5. Moreover, we used the homotopy operator to derive an extended Peach–Köhler equation and the Golebiewska gauge. These results lead us to a simple conclusion: certain kinds of equations correspond to certain kinds of operators.

In section 4, we consider basic quantities in the CTD such as Burgers vectors and Frank vectors. It is known that electric charges and Burgers vectors have similar structures [18, 22]. For instance, Burgers vectors are given by integrals of dislocation fields (see (16)) just as electric charges are given as integrals of electric fields [18]. In the monopole theory, not only electric charges but also magnetic charges exist [23]. Electric and magnetic fields can be replaced by using Hodge star operators [37]. On the other hand, the strain space, in which the Burgers vectors are defined, is also replaced by the stress space by using Hodge star operators [15]. Therefore, it is found that the magnetic charges correspond to basic quantities in the stress space such as stress functions (see (18) and (19)). Because the quantities in the stress space–time are invisible (table 1), we cannot directly observe the basic quantities in the stress space.

In section 7, we showed that T^i and $R^i{}_j \wedge \psi^j - \Gamma^i{}_j \wedge T^j$ are invariant under the transformations (36). By application of this result to the strain and stress space–times, we obtained the Golebiewska gauge transformations which were first derived by Edelen in the strain space–time [20]. Kinematic and continuity equations in the strain space–time (or stress space–time) are invariant under the Golebiewska gauge transformations in the strain space–time (or stress space–time) (see (37) and (38)). On the other hand, the constitutive equations indicate that quantities in the strain space–time (or stress space–time) are not invariant under the Golebiewska gauge transformation in its dual space–time, i.e., the stress space–time (or strain space–time) (see (39) and (40)). According to Kondo and Minagawa [24], transformations of the type shown by (39) can be physically interpreted as the criterion for yielding. Moreover, transformations of the type shown by (40) can be physically interpreted as the conditions for fatigue fracture. These results suggest that yielding and fatigue fracture are related by the Hodge duality.

In order to derive the transformations (36), we showed that T^i and $R^i{}_j \wedge \psi^j - \Gamma^i{}_j \wedge T^j$ vanish under the conditions $\psi^i = \psi^i_e - \mathrm{H}(\Gamma^i{}_j \wedge \psi^j)$ and $\Gamma^i{}_j \wedge \psi^j = (\Gamma^i{}_j \wedge \psi^j)_e$. From (30) and the definitions $\mathrm{H}B^i = u^i$, $\mathrm{H}K^i = r^i$, the relations can be rewritten physically: disclination and dislocation fields vanish under the conditions

$$B^i = \mathrm{d}u^i - r^i \qquad \text{and} \qquad K^i = \mathrm{d}r^i. \tag{41}$$

The first relation means that total distortions are given by sums of two terms: the gradient of a displacement and an internal rotation. Now, a similar equation has already been derived in the TBH model and has been used to study various plastic phenomena within polycrystals, such as lattice preferred orientation [25, 26]. The basic equation of the TBH model is

$$B^T = \sum_h^n (\mathrm{d}u^h) + r^a \tag{42}$$

where B^T is the total distortion, u^h gives the displacement field due to each slip system h and n is the number of slip systems. r^a is called the additional rotation, causing the crystal lattice to rotate within a polycrystal [26]. Recall that relation (41) was introduced in the first transformation of (36). This implies that the basic relation in the TBH model is another expression of the Golebiewska gauge in the strain space–time.

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